On the area operators of the Husain-Kuchař-Rovelli model and Canonical/Loop Quantum Gravity

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ABSTRACT

I investigate the relation between an operative definition of the area of a surface specified by matter fields and the area operators recently introduced in the canonical/loop approach to Quantum Gravity and in Rovelli's variant of the Husain-Kuchař Quantum-Gravity toy model. The results suggest that the discreteness of the spectra of the area operators might not be observable.

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One of the most intriguing aspects of Canonical/Loop Quantum Gravity [1, 2, 3] is its area operator [4] which has discrete eigenvalues. Although in the context considered in Ref. [4] areas are not diffeomorphism-invariant, the analysis reported by Rovelli in Ref. [5] suggests that a discrete spectrum should also characterize areas specified in a diffeomorphism-invariant manner [6, 7] by matter fields. In fact, in Ref. [5] this discreteness was analyzed within the model obtained by introducing matter fields in the Husain-Kuchař quantum-gravity toy model [8], whose area operator is completely analogous to the one of Canonical/Loop Quantum Gravity. In this brief note, I shall assume that indeed such a discrete area operator correctly describes areas in the quantum-gravity formalism, and investigate how its mathematical properties would affect the outcome of experiments in which areas are measured. This issue was only very briefly considered in Ref. [5].

Some of the points made in the following are relevant for the study of any diffeomorphism-invariant area operator (whether or not the spectrum is discrete). Other aspects of the analysis apply only to diffeomorphism-invariant area operators with discrete spectrum, but still the details of the spectrum are never important for the line of argument here proposed. For simplicity, the reader can assume that the area operator has eigenvalues $\mathcal{A}_n$ given by half-integer multiples of the square of the "Planck length" L_P ($L_P \sim 10^{-33} cm$):

$$\mathcal{A}_n = \frac{n}{2} L_P^2 \ , \tag{1}$$

which is the type of quantization found [5] in the Husain-Kuchař-Rovelli model.

Of course, it will be here necessary to analyze a procedure for the measurement of areas. I shall consider the procedure proposed by Rovelli in Ref. [5]. There, for simplicity, the matter fields that specify the surface whose area is being measured are taken to form a metal plate, and the area $\mathcal{A}$ of this metal plate is measured using an electromagnetic device that keeps a second metal plate at a small distance d and measures the capacity C of the so formed capacitor. Of course, measuring d and C, and assuming that $d \ll \sqrt{\mathcal{A}}$, one also measures $\mathcal{A}$ as

$$\mathcal{A} = C d \ , \tag{2}$$

where I chose for simplicity units in which the relevant permittivity is 1.

In a conventional Quantum Mechanics context one can establish with total accuracy the properties of the spectrum of a given operator in the limit in which the devices composing the measuring apparatus "behave classically." In fact, at the price of renouncing any information on a conjugate operator, in this classical-device limit (Copenhagen interpretation) one can in principle measure any given observable with total accuracy (see, *e.g.*, Ref. [9]). In such a limit one would for example uncover the nature of the discrete spectra of the area operators of interest here. In this note I shall investigate the implications for the measurability of the properties of discrete area operators of the fact that, as already observed in Ref. [10], the classical-device limit is not consistent with the nature of the gravitational interactions.

In order to illustrate in which sense the classical-device limit is not available in Quantum Gravity it is useful to briefly review the analysis reported in Ref. [10], which focused on the measurability of the distance L between (the centers of mass of) two bodies.[1] In Ref. [10] the distance L is measured via the Wigner measurement

[1]This length observable is of course diffeomorphism-invariant since the two bodies physically identify the points whose distance is being measured.

procedure [9, 11], which relies on the exchange of a signal between the two bodies. The setup of the measuring apparatus schematically requires *attaching*[2] a light-gun (*i.e.* a device capable of sending a signal when triggered), a clock, and a detector to one of the bodies and *attaching* a mirror to the other body. By measuring the time T needed by the signal for a two-way journey between the bodies one also obtains a measurement of L. [For example, in Minkowski space and neglecting quantum effects one simply finds that $L = cT/2$, with c denoting the speed of light.] Within this setup it is easy to realize that δL can vanish only if all devices used in the measurement behave classically. One can consider for example the contribution to δL coming from the uncertainties that affect the relative motion of the clock with respect to the center of mass of the system composed by the light-gun and the detector. This relative position is crucial for the measurement procedure since it is associated to two time delays that must be taken into account in extracting a measurement of L. The first time delay occurs initially when the clock triggers the light-gun and the second time delay occurs in the end when (having collected the "return signal") the detector stops the clock. Let us denote with x^* the relative position of the clock with respect to the center of mass of the system composed by light-gun and detector, and use v^* to denote the corresponding relative velocity. It is easy to show [9, 10, 11] that the uncertainties δx^* and δv^* that characterize the state in which the experimentalist prepares the devices contribute to δL according to

$$\delta L \geq \delta x^* + T \delta v^* \geq \delta x^* + \frac{(M_c + M_{l+d})}{2 M_c \, M_{l+d}} \frac{\hbar T}{\delta x^*} \ , \tag{3}$$

where M_c is the mass of the clock, M_{l+d} is the total mass of the system composed of the light-gun and the detector, and the right-hand-side relation follows from observing that Heisenberg's *Uncertainty Principle* implies $\delta x^* \delta v^* \geq \hbar (M_c + M_d)/(2 M_c M_d)$. Clearly, Eq. (3) implies that $\delta L = 0$ can only be achieved in the "classical-device limit," understood as the limit of infinitely large M_c and M_{l+d}. This is consistent with the nature of the ordinary Quantum-Mechanics framework, which relies on classical devices. However, once gravitational interactions are taken into account the classical-device limit is no longer available. Large values of the masses M_c and M_{l+d} necessarily lead to great distorsions of the geometry, and well before the $M_c, M_{l+d} \to \infty$ limit the Wigner measurement procedure can no longer be completed. [For large enough masses we even expect that "information walls" (the ones of black-hole physics) would form between the elements of the measurement procedure.]

Since the classical limit $M_c, M_{l+d} \to \infty$ is not available, from Eq.(3) one concludes that in Quantum Gravity the uncertainty on the measurement of a length grows with the time T required by the measurement procedure (as it happens in presence of decoherence effects [12]). In fact, from Eq.(3) one arrives [10] at a minimum uncertainty for the measurement of a distance L of the type[3]

$$minimum \, [\delta L] \, \sim \, \sqrt{cT L^*_{QG}} \, \sim \, \sqrt{L \, L^*_{QG}} \ , \tag{4}$$

[2]Of course, for consistency with causality, in such contexts one assumes devices to be "attached non-rigidly," and, in particular, the relative position and velocity of their centers of mass satisfy the standard uncertainty relations of Quantum Mechanics.

[3]Besides the uncertainties introduced by the devices there should also be a measurement-procedure-independent contribution L_{QG} to this uncertainty. In most Quantum-Gravity scenarios L_{QG} is identified with the Planck length [13], whereas in String Theory L_{QG} is the string length [14]. I am here for simplicity not keeping track of this "minimum length," whose implications (possibly involving non-locality [15]) are by now well accepted. From the results of Refs. [10, 16], where the

where L^*_{QG} is a Quantum-Gravity length scale that characterizes the above-mentioned limitations due to the absence of classical devices, and the relation on the right-hand side follows from the fact that T is naturally proportional [10, 11] to L. Although L^*_{QG} emerges in a way that does not appear to be directly related to the Planck length, it seems plausible [10] that $L^*_{QG} \sim L_P$.

Having clarified in which sense classical devices are not available in Quantum Gravity, and having briefly reviewed how this affects the measurability of distances, we can now return to the analysis of the limitations on the measurability of the properties of discrete area operators. According to Eq. (2), in general the uncertainty in the measurement of the area $\mathcal{A}$ receives contributions from uncertainties in the determination of C and d. Since I am aiming for a final result formulated as a measurability bound (*i.e.* a lower bound on the uncertainty), it is legitimate to ignore the contribution coming from the uncertainty in C and focus on the contribution coming from the uncertainty in d

$$\delta\mathcal{A} \,\geq\, C\,\delta d \,=\, \frac{\delta d}{d}\,\mathcal{A}\,, \tag{5}$$

where I also used again Eq. (2) to eliminate C.

Based on the bound (4) on the measurability of distances one can assume that $\delta d/d \geq \sqrt{L^*_{QG}/d}$ and therefore

$$\delta\mathcal{A} \,\geq\, \sqrt{L^*_{QG}}\,\frac{\mathcal{A}}{\sqrt{d}}\,. \tag{6}$$

This relation confronts us with a scenario similar to the one of Eq. (3). It formally admits a limit ($d \to \infty$) in which the area could be measured with complete accuracy, but this limit cannot be reached within the constraints set by the nature of the measurement procedure. In fact, the relation (2), on which the measurement procedure is based, only holds for $d \ll \sqrt{\mathcal{A}}$, and in considering larger and larger d one quickly ends up loosing all information on $\mathcal{A}$. A rather safe lower bound is therefore obtained by imposing $d \leq \sqrt{\mathcal{A}}$ in Eq. (6), which gives

$$\delta\mathcal{A} \,\geq\, \sqrt{L^*_{QG}}\,\mathcal{A}^{3/4}\,. \tag{7}$$

Interestingly, Eq. (7) is, like Eq. (4), the result of the uncertainties in the position of a device, in this case the metal plate used for the measurement. However, Eq. (7) was not derived by observing that the limit of infinitely-massive metal plate is not available. Rather than resulting from the properties of the metal plate, the bound (7) follows from the general limitations on the measurability of distances encoded in Eq. (4). Of course, a more detailed analysis of this measurement procedure would have to take into account also the properties of the metal plate. While I shall not here attempt such a delicate analysis, it is perhaps worth emphasizing some of its elements of difficulty. The fact that ideal measuring plates (just like ideal measuring

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measurability of distances was discussed taking into account both the uncertainty introduced by the "non-classical" devices and the uncertainty associated to the minimum length, the reader can easily realize that L_{QG} would not affect the line of argument here presented.

rods [9, 10, 11]) would not be available in Quantum Gravity can be discussed very simply by viewing a plate as composed by elementary cells, possibly of size $L_P^2/2$ as encoded in area-quantization relations of the type (1). In ordinary (non-quantum) gravitational contexts one would obtain such a plate by setting in rigid motion the elementary cells that compose it; however, once quantum effects are switched on the *Uncertainty Principle* does not allow to maintain fixed during the measurement the relative position of the elementary cells composing the plate, thereby excluding the possibility of rigid motion. This Quantum-Gravity description[4] of a plate of course introduces new elements of uncertainty in the analysis of the measurability of areas measured using the procedure considered in the present note. Having ignored this (which is probably the most significant) source of limitations for the measurability, we can expect that the actual measurability bound for areas in Quantum Gravity could be much tighter than the bound (7).

While even tighter measurability bounds might be uncovered by more refined analyses, already the bound (7) appears to require a significant shift in the physical interpretation of area-quantization relations of the type (1). In Ref. [5] it was observed that the formal property (1) of the area operator in the Hilbert space of the Husain-Kuchař-Rovelli model (and similar considerations should apply to the area operator of Canonical/Loop Quantum Gravity) would directly affect the outcome of area mesurements within the conventional Quantum Mechanics framework, *i.e.* the outcome of area measurements should be $L_P^2/2$ quantized. I have here observed that the conventional Quantum Mechanics framework, with its classical measuring apparatus, is inconsistent with the nature of gravitational interactions and that the heuristic analysis of a new Quantum Gravity measurement framework appears to suggest that the area quantization encoded in the formalism *might not be observable.* In fact, assuming $L_{QG}^* \sim L_P$, Eq. (7) indicates that the measurement of a given area of order $nL_P^2/2$ would be affected by an uncertainty of at least $\sim L_P^2(n/2)^{3/4}$, *i.e.* (for every area with $n > 1$) an uncertainty much larger than the $L_P^2/2$ quanta.

Concerning the physical interpretation of Eq. (7) one is also naturally led to inquire about the type of symmetries that could result in such a structure. Of course, it will be possible to rigorously address this question only once a formalism supporting relations such as (4) and (7) is found; however, some consistency arguments [16, 17] appear to indicate that dimensionful deformations of Poincaré symmetries might be involved. While I shall not repeat those arguments here, it is worth emphasizing, as a reason of interest in the scenario advocated in the present note, that such deformations of Poincaré symmetries could soon be tested [18] experimentally by exploiting the recent dramatic developments in the phenomenology of gamma-ray bursts [19].

In closing, let me summarize the points made in this note also clarifying which ones could be considered as "robust." The way in which the new bound (7) has been here derived involves rather heuristic arguments, and might reflect the structure of the specific example of procedure for the measurement of areas which has been considered.

[4]It is perhaps worth emphasizing that ideal measuring plates (like other ideal classical devices) are consistent with the laws of ordinary (non-gravitational) Quantum Mechanics. In fact, in the limit in which each elementary cell has infinite mass the *Uncertainty Principle* ceases to affect the dynamics of the cells and therefore the cells can (at least in principle) be set in rigid motion with respect to one another. This infinite-mass classical limit is perfectly consistent with the conceptual structure of Quantum Mechanics and with the non-gravitational analysis of measurement procedures, but, as explained above, it is not consitent with the nature of measurement procedures once gravitational interactions are turned on.

Accordingly, Eq. (7) is to be considered as very preliminary, and in particular more refined analyses might find that the $\mathcal{A}$-dependence on the right-hand-side comes in with an exponent different from 3/4. However, Eq. (7) should be expected to capture the correct qualitative behavior, *i.e.* a limitation on area measurability that grows with the size of the area. In fact, such a behavior already characterizes measurements in ordinary Quantum Mechanics, unless the infinite-mass "classical-device" limit is taken. The observation that this "classical-device" limit is not available once gravitational interactions are taken into account is nearly self-evident and has been here discussed rather intuitively. I emphasize that this observation is not in contradiction with the point made by some authors (see, *e.g.*, Ref. [5]) that even in Quantum Gravity the measuring apparatus should be "external" to the system under observation. Indeed, I have analyzed the measuring devices as external to the system, *i.e.* I maintained at all stages the distinction between the degrees of freedom of the system being observed and the ones of the measuring apparatus. The novel element of the analysis here reported is that I refrained from assuming that these external devices would somehow not be subject to the same laws of physics that govern the dynamics of the system under observation. This assumption is not made in ordinary non-gravitational Quantum Mechanics[5] and there appears to be no reason why it should be made in the Quantum-Gravity context. Assuming this point is correct Eq. (7) should capture the correct qualitative structure of the area measurability bound in Quantum Gravity, and consequently, as explained above, one expects that the output of area measurements could not take the form of integer multiples of the $L_P^2/2$ quanta even though such a quantization charaterizes the formal spectrum of the area operator.

Besides considering other procedures for the measurement of areas and refining the analysis of the implications of the non-classical behavior of devices, future work aiming at establishing more precisely the form of the area measurability bound in Quantum Gravity should also consider how the *Equivalence Principle* could affect the measurement procedures. This might play a crucial role in the way the devices interact with the system being observed. The conceptual framework of ordinary Quantum Mechanics relies not only on the infinite-mass "classical-device" limit, but also on the limit in which the devices decouple from the system. For example the devices used in the measurement of the electromagnetic field interact with it, but (as emphasized in Refs. [16, 20, 21]) in measurability analyses within ordinary Quantum Mechanics it is crucial that there is a limit in which devices fully decouple from the field. In the case of the electromagnetic field this limit is the one in which the devices have vanishing ratio of electric charge to inertial mass. Of course, the devices of a Quantum-Gravity apparatus also interact with the gravitational field, and the limit in which the devices decouple from the field appears not to be available as a result of the fact that the ratio of gravitational charge to inertial mass is fixed by the *Equivalence Principle*. This has not played a role in the present analysis, but could be important in more refined studies of measurability of area observables or other Quantum Gravity observables.

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[5] As illustrated by some of the points made in this note (*e.g.* the different descriptions that Quantum Gravity and ordinary Quantum Mechanics give of a measuring plate), in ordinary non-gravitational Quantum Mechanics by considering measuring devices that behave classically one is not actually assuming that the laws of Quantum Mechanics would not apply to these devices; in fact, the classical limit (typically involving an infinite-mass limit) is perfectly well defined within ordinary Quantum Mechanics and leads to no pathologies as long as the gravitational interactions are ignored.

References

[1] A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244; Phys. Rev. D36 (1987) 1587.

[2] C. Rovelli and L. Smolin, Phys. Rev. Lett. 61 (1988) 1155; Nucl. Phys. B331 (1990) 80.

[3] A. Ashtekar and C.J. Isham, Class. Quantum Grav. 9 (1992) 1433.

[4] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. 69 (1992) 237; C. Rovelli, and L. Smolin, Nucl. Phys. B442 (1995) 593; A. Ashtekar and J. Lewandowski, Class. Quantum Grav. 14 (1997) A55.

[5] C. Rovelli, Nuc. Phys. B405 (1993) 797.

[6] A. Einstein, *Über die spezielle und allgemeine Relativitätstheorie* (Vieweg, Braunscheig, 1920); P.G. Bergmann, Helv. Phys. Acta Suppl. 4 (1956) 79; B.S. De Witt, in *Gravitation: An introduction to recent research*, ed. L. Witten (Wiley, New York, 1962); J. Stachel, in *Einstein and the history of General Relativity*, ed. D. Howard and J. Stachel (Birkhäuser, Boston, 1989).

[7] C. Rovelli, Class. Quantum Grav. 8 (1991) 297; *ibid.* 8 (1991) 317.

[8] V. Husain and K. Kuchař, Phys. Rev. D42 (1990) 4070.

[9] E.P. Wigner, Rev. Mod. Phys. 29, 255 (1957); H. Salecker and E.P. Wigner, Phys. Rev. 109, 571 (1958).

[10] G. Amelino-Camelia, Mod. Phys. Lett. A9, 3415 (1994); Mod. Phys. Lett. A11 (1996) 1411.

[11] Y.J. Ng and H. Van Dam, Mod. Phys. Lett. A9, 335 (1994); Mod. Phys. Lett. A10, 2801 (1995).

[12] Karolyhazy, Il Nuovo Cimento A42 (1966) 390; L. Diosi, B. Lukacs, KFKI-1989-52-B.

[13] See, *e.g.*, T. Padmanabhan, Class. Quantum Grav. 4 (1987), L107; M. Maggiore, Phys. Lett. B304 (1993), 65; S. Doplicher, K. Fredenhagen, and J.E. Roberts, Phys. Lett. B331 (1994), 39; L.J. Garay, Int. J. Mod. Phys. A10 (1995), 145; A. Kempf, G. Mangano, and R.B. Mann, Phys. Rev. D52 (1995), 1108.

[14] D.J. Gross and P.F. Mende, Nucl. Phys. B303 (1988) 407; D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B216 (1989), 41; K. Konishi, G. Paffuti, P. Provero, Phys. Lett. B234 (1990), 276.

[15] D.V. Ahluwalia, Phys. Lett. B339 (1994) 301.

[16] G. Amelino-Camelia, "Dimensionful deformations of Poincaré symmetries for a Quantum Gravity without ideal observers" (in preparation).

[17] G. Amelino-Camelia, Phys. Lett. B392 (1997) 283; G. Amelino-Camelia, J. Lukierski, and A. Nowicki, hep-th/9706031.

[18] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, S. Sarkar, astro-ph/9712103, *Nature* (in press).

[19] For a recent review, see *e.g.* M. Livio et al., astro-ph/9712097, to appear in *Proceedings of the 4th Huntsville Gamma-Ray Burst Symposium*.

[20] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab S. Nat. Fys. Medd. 12 (1933) 1.

[21] P.G. Bergmann and G.J. Smith, Gen. Rel. Grav. 14 (1982) 1167.